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EFFECT OF COMPRESSIBILITY AT HIGH SUBSONIC VELOCITIES ON THE MOMENT ACTING ON AN ELLIPTIC CYLINDER

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SUMMARY

An extended form of the Ackeret iteration process is utilized to calculate the compressible flow at high subsonic velocities past an elliptic cylinder. The angle of attack with respect to the direction of the undisturbed stream is assumed small and the circulation is fixed by the condition that the trailing end of the major axis be a stagnation point. The expression for the moment acting on the elliptic cylinder is derived and shows a first-step improvement of the Prandtl-Glauert approximation. In addition, a second-step improvement is obtained in the Prandtl-Glauert approximation for the lifting force acting on the elliptic cylinder. By means of these two results it is possible to calculate the effect of compressibility on the position of the center of pressure as a function of the thickness coefficient and of the stream Mach number. Tables and corresponding graphs are included to illustrate numerically the theoretical results derived. For example, it is found that, for an elliptic profile of thickness coefficient 0.15 and stream Mach number 0.80, the center of pressure moves rearward a distance 2.6 percent of the chord from its position in the incompressible flow.

INTRODUCTION

The present paper is concerned mainly with the calculation of the effect of compressibility at high subsonic velocities on the moment acting on an elliptic cylinder. The method used is an iteration procedure, credited to Ackeret, which proceeds from the Prandtl-Glauert approximation as the first step and successively improves it in a systematic manner. The details of the Ackeret iteration process have been described in reference 1 and, therefore, only material essential to the present paper will be repeated.

The main purpose of the Ackeret iteration method is to linearize the nonlinear partial differential equation (for the velocity potential or the stream function) that governs the steady two-dimensional flow of a perfect compressible fluid. This linearization is accomplished by assuming the development of the stream function ψ, say, to be of the form

$$\psi = -UY + \psi_1(X,Y) + \psi_2(X,Y) + \psi_3(X,Y) + \ldots \tag{1}$$

where U is the velocity of the undisturbed stream and X and Y are the rectangular Cartesian coordinates of the physical flow plane. Equation (1) is essentially a development of the stream function around a uniform stream in the negative direction of the X-axis. For the purpose of defining or controlling the iteration procedure, the function ψ_{n+1} is regarded as small compared with the preceding function ψ_n and the derivatives have a similar relationship. Then the total index decides the order of the term; for example, ψ_3 is of the same order as ${\psi_1}^3$ or $\psi_1\psi_2$. The accuracy of this iteration method clearly depends on the degree to which the assumptions are satisfied. In the case of slender bodies without stagnation points, the first few steps may be expected to yield a good result. In the case of bodies with stagnation points, the accuracy of the calculations obviously depends on the number of terms ψ_n derived, each new term reducing the extent of the region of inaccuracy in the neighborhood of the stagnation point.

In the treatment of the various equations that result from the linearization of the fundamental differential equation by means of the Ackeret iteration process, it is convenient to introduce an affine transformation of the physical flow plane. This affine transformation reduces the differential equations to be solved to a Laplace equation and to Poisson equations. In the performance of this simplification, the statement of the boundary condition at the solid by means of the velocity potential becomes very complicated. Fortunately, however, the statement of the boundary condition by means of the stream function, namely, $\psi = 0$ at the solid, is invariant for the affine transformation; therefore, the use of the stream function throughout the analysis of the present paper is to be preferred. The choice of the ellipse as the solid boundary is dictated by the property that an affine distortion of an ellipse leads to another ellipse; therefore, the analysis can be conducted entirely in the affinely distorted plane and the results thus obtained linked to the actual elliptic profile by means of simple correspondence relations.

MOMENT FORMULA

Specifically, the problem treated herein is to obtain an improvement of the Prandtl-Glauert approximation of the effect of compressibility on the moment acting on an elliptic cylinder set at a small angle of attack in a uniform stream. Let Z denote the physical flow plane, z the affinely transformed plane, and z' the plane of the circle into which the affinely distorted profile is mapped by a conformal transformation. (See fig. 1.) As in the calculation of the resultant lifting force given in reference 1, it is a great labor saving device to choose a large circle in the z'-plane to correspond to the control contour in the physical Z-plane during the calculation of the moment and also to choose as independent variables the polar coordinates $Re^{\eta}, -\xi$ of the z'-plane, with

$$z' = Re^{-i\zeta} \tag{2}$$

where $\zeta = \xi + i\eta$ and R is the radius of the conformal circle. (See fig. 1(c).)

Since the large circle in the z'-plane corresponds to a large control ellipse in the physical flow plane Z, the expression for the moment must contain, in addition to the usual momentum integral, a term involving the integration of the pressures around the control ellipse. This additional term is necessary because the normal vector to an ellipse does not pass through its center. The general vector expression for the moment in a compressible fluid M_c with respect to the origin, obtained from reference 2, is

$$M_c = \oint \rho \left[\bar{r}\bar{q}\right]\left(\bar{q}\bar{n}\right) ds + \oint \left[\bar{r}\bar{n}\right] p \, ds \tag{3}$$

where brackets and parentheses denote vector and scalar products, respectively, and

$\bar{r}$ radius vector from origin

$\bar{n}$ unit normal vector

$\bar{q}$ velocity vector of fluid

ds element of length along control contour

p pressure of fluid

ρ density of fluid

The positive direction of the unit normal vector $\bar{n}$ is from the control contour toward the origin, and the line integrals are taken positively counterclockwise around the control contour in the physical flow plane. The unit tangent vector $\bar{t}$ and the unit normal vector $\bar{n}$ thus form a right-hand frame; hence, a positive value for the moment corresponds to a counterclockwise rotation (fig. 1(a)).

It is easy to verify that equation (3) can be rewritten in the form

$$M_c = -\frac{1}{2}\text{R.P.}\oint \rho(u - iv)^2 Z\, dZ + \frac{1}{2}\oint\left(p + \frac{1}{2}\rho q^2\right) dr^2 \qquad (4)$$

where u and v are the components of the velocity vector along the X-axis and Y-axis, respectively, and

$$Z = X + iY$$

$$r^2 = Z\bar{Z} = X^2 + Y^2$$

$$q^2 = u^2 + v^2$$

Note that if the fluid is incompressible

$$p + \frac{1}{2}\rho q^2 = \text{Constant}$$

according to Bernoulli's equation; therefore, the second integral vanishes identically and yields the usual Blasius formula for the moment. By use of the adiabatic relation $\frac{p}{p_1} = \left(\frac{\rho}{\rho_1}\right)^{\gamma}$, equation (4) becomes

$$M_c = -\frac{1}{2}\rho_1 U^2 R.P.\oint \frac{\rho_1}{\rho}\left[\frac{\rho}{\rho_1 U}(u - iv)\right]^2 Z\, dZ + \frac{1}{2}\rho_1 U^2 \oint \left[\frac{1}{\gamma M_1{}^2}\left(\frac{\rho_1}{\rho}\right)^{-\gamma} + \frac{1}{2}\frac{\rho_1}{\rho}\left(\frac{\rho q}{\rho_1 U}\right)^2\right] dr^2 \qquad (5)$$

where the subscript 1 refers to the starting conditions at infinity and

U velocity of undisturbed fluid at infinity

M_1 Mach number of undisturbed stream at infinity (U/c_1)

c_1 velocity of sound in undisturbed fluid

γ ratio of specific heats at constant pressure and constant volume, for air $\gamma = 1.4$

For the purpose of calculating the line integrals indicated in equation (5), it is necessary to express the integrands as functions of the independent variables ξ,η of the z'-plane. In the case of the elliptic profile in the z-plane, the conformal transformation to a circle of radius R with center at the origin is

$$z = c \cos(\zeta + i\lambda) \qquad (6)$$

where c is the semifocal distance and λ is defined by any one of the following geometric characteristics of the ellipse:

$$a = c \cosh \lambda$$

$$b = c \sinh \lambda$$

$$R = \frac{1}{2} c e^{\lambda}$$

where a, b, and R are, respectively, semimajor axis of ellipse, semiminor axis of ellipse, and radius of conformal circle. Now, the affine transformation used in connection with the Ackeret iteration process is

$$X = x$$

$$Y = \mu y$$

where

$$\mu = \frac{1}{\sqrt{1 - M_1{}^2}}$$

It follows that

$$Z = \frac{1 + \mu}{2} z + \frac{1 - \mu}{2} \bar{z}$$

and

$$dZ = \frac{1 + \mu}{2} dz + \frac{1 - \mu}{2} d\bar{z}$$

where a bar indicates conjugate-complex quantities. Since the control contour is a large circle in the z'-plane, η = Constant and $d\zeta = d\bar{\zeta} = d\xi$. Then, by use of equations (2) and (6) and the relation $R = \frac{1}{2} c e^{\lambda}$, the expressions for Z and dZ on the control ellipse become

$$Z = \frac{ce^{\eta+\lambda}}{4}\left[(1 + \mu)\left(\frac{1}{z'e^{2\eta+2\lambda}} + z'\right) + (1 - \mu)\left(\frac{1}{z'} + \frac{z'}{e^{2\eta+2\lambda}}\right)\right] \qquad (7a)$$

and

$$dZ = -\frac{ce^{\eta+\lambda}}{4}\left[(1 + \mu)\left(\frac{1}{z'e^{2\eta+2\lambda}} - z'\right) + (1 - \mu)\left(\frac{1}{z'} - \frac{z'}{e^{2\eta+2\lambda}}\right)\right]\frac{dz'}{z'} \qquad (7b)$$

where $z' = e^{-i\xi}$

Similarly,

$$r^2 = Z\overline{Z} = \frac{1 - \mu^2}{4}z^2 + \frac{1 - \mu^2}{4}\overline{z}^2 + \frac{1 + \mu^2}{2}z\overline{z}$$

and on the control ellipse, η = Constant,

$$dr^2 = \frac{c^2e^{2\eta+2\lambda}}{8}\left[\left(\mu^2 - 1\right)\left(\frac{1}{z'^2e^{4\eta+4\lambda}} - z'^2\right) + \left(\mu^2 - 1\right)\left(\frac{1}{z'^2} - \frac{z'^2}{e^{4\eta+4\lambda}}\right)\right.$$

$$\left. + 2\left(\mu^2 + 1\right)e^{-2\eta-2\lambda}\left(z'^2 - \frac{1}{z'^2}\right)\right]\frac{dz'}{z'} \qquad (8)$$

Expressions for $\frac{\rho}{\rho_1 U}(u - iv)$ and ρ_1/ρ as functions of the variable $z'\left(= e^{-i\xi}\right)$, expanded in powers of $1/e^{\eta}$, are given by equations (46) and (47), respectively, of reference 1. By the use of these equations, together with equations (7) and (8), it is easy to evaluate the right-hand member of equation (5) by noting that only terms involving dz'/z' contribute to the line integrals. The result thus obtained is

$$M_c = \pi\rho_1 U^2 \alpha \frac{(a+b)^2}{8} \left\{ 8\mu^2 \frac{a-b}{a+b} - \mu^2\left(\mu^2 - 1\right)\left[8 + \left(\mu^2 + 1\right)(\sigma + 4)\right]\left(\frac{b}{a+b}\right)^2 \right\}$$

where α is the angle of attack in the affinely distorted plane z and $\sigma = (\gamma + 1)\left(\mu^2 - 1\right)$. If the quantities a, b, c, and α are replaced by a', b', c', and α' of the actual profile in the Z-plane according to the correspondence equations of reference 1, namely,

$$\left.\begin{aligned} a &= a' \\ b &= \frac{1}{\mu} b' \\ c^2 &= c'^2 + \frac{\mu^2 - 1}{\mu^2} b'^2 \\ \alpha &= \frac{1}{\mu} \alpha' \end{aligned}\right\} \tag{9}$$

the moment about the origin on the actual ellipse becomes

$$M_c = \pi\rho_1 U^2 \alpha' c'^2 \mu - \frac{1}{8}\pi\rho_1 U^2 \alpha' b'^2 (\sigma + 4)\frac{\mu^4 - 1}{\mu} \tag{10}$$

Now, for an incompressible fluid,

$$M_1 = 0$$

or

$$\mu = 1$$

and

$$M_1 = \pi\rho_1 U^2 \alpha' c'^2$$

Therefore,

$$\frac{M_c}{M_1} = \mu - \frac{1}{8\mu}(\sigma + 4)\left(\mu^4 - 1\right)\frac{t'^2}{1 - t'^2} \tag{11}$$

where t' is the thickness coefficient b'/a' of the actual elliptic profile in the physical flow plane.

Equation (11) represents a first-step improvement of the Prandtl-Glauert approximation and reduces to that result in the limiting case $t' \longrightarrow 0$. This improvement, however, is incomplete, for, as can be observed from equation (10), the second term on the right-hand side is of the third order (that is, proportional to $\alpha' b'^2$) and terms of that order are contributed mainly by ψ_3. Since only the first two terms ψ_1 and ψ_2 were derived in reference 1, it is necessary to determine the third term ψ_3 in order to obtain the complete first-step improvement of the Prandtl-Glauert approximation for M_c/M_1.

DETERMINATION OF ψ_3

In order to obtain the third term ψ_3 in the expansion for the stream function

$$\psi = -UY + \psi_1 + \psi_2 + \psi_3 + \dots \tag{12}$$

it is first necessary to obtain the expression for ρ_1/ρ, inclusive of third order terms, in the neighborhood of the undisturbed stream. Analogous to equation (19) of reference 1, this expression is

$$\frac{\rho_1}{\rho} = 1 - \left(\mu^2 - 1\right)\frac{\psi_{1Y}}{U} - \left(\mu^2 - 1\right)\left(\frac{\psi_{2Y}}{U} - \frac{\psi_{1X}^2 + \psi_{1Y}^2}{2U^2}\right) + \frac{1}{2}\left(\mu^2 - 1\right)^2\left[(\gamma + 4) + (\gamma + 1)\left(\mu^2 - 1\right)\right]\frac{\psi_{1Y}^2}{U^2}$$

$$+ \left(\mu^2 - 1\right)\left(\frac{\psi_{1X}\psi_{2X} + \psi_{1Y}\psi_{2Y}}{U^2} - \frac{\psi_{3Y}}{U}\right) + \left(\mu^2 - 1\right)^2\left[(\gamma+4) + (\gamma + 1)\left(\mu^2 - 1\right)\right]\left(\frac{\psi_{1Y}\psi_{2Y}}{U^2} - \psi_{1Y}\frac{\psi_{1X}^2 + \psi_{1Y}^2}{2U^3}\right)$$

$$- \frac{1}{6}\left(\mu^2 - 1\right)^3\left\{15 + (\gamma + 1)\mu^2\left[(2\gamma + 15) + 3(\gamma + 1)\left(\mu^2 - 1\right)\right]\right\}\frac{\psi_{1Y}^3}{U^3} + \ldots \tag{13}$$

When the expressions for ψ and ρ_1/ρ given by equations (12) and (13) are substituted into the basic differential equation

$$\frac{\partial}{\partial X}\left(\frac{\rho_1}{\rho}\frac{\partial\psi}{\partial X}\right) + \frac{\partial}{\partial Y}\left(\frac{\rho_1}{\rho}\frac{\partial\psi}{\partial Y}\right) = 0 \tag{14}$$

and terms of the third order in the derivatives of ψ_n are collected, the following differential equation for ψ_3 is obtained:

$$
\begin{aligned}
\psi_{3XX} + \mu^2\psi_{3YY} = {} & 2\left(\mu^2 - 1\right)\frac{\psi_{1X}\psi_{2XY} + \psi_{1XY}\psi_{2X}}{U} + \left[\frac{1}{2}(\gamma + 1)\left(\mu^2 - 1\right)^2 - \frac{\mu^2 - 1}{\mu^2}\right]\frac{{\psi_{1X}}^2}{U^2}\psi_{1XX} \\
& - \left(\mu^2 - 1\right)\left[2 + (\gamma + 1)\left(\mu^2 - 1\right)\right]\frac{\psi_{1Y}\psi_{2XX} + \psi_{1XX}\psi_{2Y}}{U} - 2\left(\mu^2 - 1\right)\left[1 + (\gamma + 1)\left(\mu^2 - 1\right)\right]\frac{\psi_{1X}\psi_{1Y}\psi_{1XY}}{U^2} \\
& + \Bigg(\frac{\mu^2 - 1}{\mu^2} + \frac{\left(\mu^2 - 1\right)^2}{2\mu^2}\left[(5\gamma + 7) - (\gamma + 1)\left(\mu^2 - 1\right)\right] + \frac{\left(\mu^2 - 1\right)^3}{2\mu^2}\Bigg\{15 + (\gamma + 1)\mu^2\left[2\gamma + 15 + 3(\gamma + 1)\left(\mu^2 - 1\right)\right] \\
& - \left[(\gamma + 4) + (\gamma + 1)\left(\mu^2 - 1\right)\right]\left[5 + 2(\gamma + 1)\left(\mu^2 - 1\right)\right]\Bigg\}\Bigg)\frac{{\psi_{1Y}}^2}{U^2}\psi_{1XX} \qquad (15)
\end{aligned}
$$

This differential equation can be expressed in a convenient form for solution by making use of the affine transformation

$$x = X$$

$$y = \frac{1}{\mu}Y$$

and by introducing a new stream function ψ^*, where

$$\psi = \mu U\psi^*$$

Then equation (15) becomes

$$
\begin{aligned}
\psi^*_{3xx} + \psi^*_{3yy} = {} & 2\left(\mu^2 - 1\right)\left(\psi^*_{1x}\psi^*_{2xy} + \psi^*_{1xy}\psi^*_{2x}\right) + \left(\mu^2 - 1\right)\left[-1 + \frac{1}{2}(\gamma + 1)\mu^2\left(\mu^2 - 1\right)\right]\psi^{*\,2}_{1x}\psi^*_{1xx} \\
& -\left(\mu^2 - 1\right)\left[2 + (\gamma + 1)\left(\mu^2 - 1\right)\right]\left(\psi^*_{1y}\psi^*_{2xx} + \psi^*_{1xx}\psi^*_{2y}\right) - 2\left(\mu^2 - 1\right)\left[1 + (\gamma + 1)\left(\mu^2 - 1\right)\right]\psi^*_{1x}\psi^*_{1y}\psi^*_{1xy} \\
& + \frac{\mu^2 - 1}{\mu^2}\Bigg(1 + \frac{1}{2}\left(\mu^2 - 1\right)\left[(5\gamma + 7) - (\gamma + 1)\left(\mu^2 - 1\right)\right] + \frac{1}{2}\left(\mu^2 - 1\right)^2 \Big\{15 + (\gamma + 1)\mu^2\left[(2\gamma + 15) + 3(\gamma + 1)\left(\mu^2 - 1\right)\right] \\
& - \left[(\gamma + 4) + (\gamma + 1)\left(\mu^2 - 1\right)\right]\left[5 + 2(\gamma + 1)\left(\mu^2 - 1\right)\right]\Big\}\Bigg)\psi^{*\,2}_{1y}\psi^*_{1xx}
\end{aligned} \tag{16}
$$

Again, as in reference 1, it will be found that the mathematical analysis will be considerably simplified by working with a nonanalytic complex potential $w^*_3(z,\bar{z})$ instead of its imaginary part ψ^*_3. As shown in reference 1, ψ^*_1 is the imaginary part of an analytic function $w^*_1(z)$, whereas ψ^*_2 is the imaginary part of a nonanalytic function $w^*_2(z,\bar{z})$. It must be emphasized that the real parts of these complex functions are not to be interpreted as velocity potentials but only as functions that render the analysis elegant and simple. The following identities can be easily verified:

$$\psi_{1x} = \frac{1}{2i}\left(w_{1z} - \bar{w}_{1\bar{z}}\right)$$

$$\psi_{1y} = \frac{1}{2}\left(w_{1z} + \bar{w}_{1\bar{z}}\right)$$

$$\psi_{1xy} = \frac{1}{2}\left(w_{1zz} + \bar{w}_{1\bar{z}\bar{z}}\right)$$

$$\psi_{1xx} = \frac{1}{2i}\left(w_{1zz} - \bar{w}_{1\bar{z}\bar{z}}\right)$$

$$\psi_{2x} = \frac{1}{2i}\left(w_{2z} + w_{2\bar{z}} - \bar{w}_{2z} - \bar{w}_{2\bar{z}}\right)$$

$$= \text{I.P.}\left(w_{2z} + w_{2\bar{z}}\right)$$

$$\psi_{2y} = \frac{1}{2}\left(w_{2z} - w_{2\bar{z}} - \bar{w}_{2z} + \bar{w}_{2\bar{z}}\right)$$

$$= \text{I.P.}\,i\left(w_{2z} - w_{2\bar{z}}\right)$$

$$\psi_{2xy} = \frac{1}{2}\left(w_{2zz} - w_{2\bar{z}\bar{z}} - \bar{w}_{2zz} + \bar{w}_{2\bar{z}\bar{z}}\right)$$

$$= \text{I.P.}\,i\left(w_{2zz} - w_{2\bar{z}\bar{z}}\right)$$

$$\psi_{2xx} = \frac{1}{2i}\left(w_{2zz} + 2w_{2z\bar{z}} + w_{2\bar{z}\bar{z}} - \bar{w}_{2zz} - 2\bar{w}_{2z\bar{z}} - \bar{w}_{2\bar{z}\bar{z}}\right)$$

$$= \text{I.P.}\left(w_{2zz} + 2w_{2z\bar{z}} + w_{2\bar{z}\bar{z}}\right)$$

where the asterisk has been dropped. Then

$$\psi_{1x}{}^2\psi_{1xx} = -\frac{1}{8i}\left(w_{1z} - \bar{w}_{1\bar{z}}\right)^2\left(w_{1zz} - \bar{w}_{1\bar{z}\bar{z}}\right)$$

$$= -\frac{1}{4}\left(w_{1z} - \bar{w}_{1\bar{z}}\right)^2 \mathrm{I.P.}\, w_{1zz}$$

$$\psi_{1y}{}^2\psi_{1xx} = \frac{1}{8i}\left(w_{1z} + \bar{w}_{1\bar{z}}\right)^2\left(w_{1zz} - \bar{w}_{1\bar{z}\bar{z}}\right)$$

$$= \frac{1}{4}\left(w_{1z} + \bar{w}_{1\bar{z}}\right)^2 \mathrm{I.P.}\, w_{1zz}$$

$$\psi_{1x}\psi_{2xy} + \psi_{1xy}\psi_{2x} = \frac{1}{2}\mathrm{I.P.}\left(w_{1z} - \bar{w}_{1\bar{z}}\right)\left(w_{2zz} - w_{2\bar{z}\bar{z}}\right)$$

$$+ \frac{1}{2}\left(w_{1zz} + \bar{w}_{1\bar{z}\bar{z}}\right)\mathrm{I.P.}\left(w_{2z} + w_{2\bar{z}}\right)$$

$$\psi_{1y}\psi_{2xx} + \psi_{1xx}\psi_{2y} = \frac{1}{2}\left(w_{1z} + \bar{w}_{1\bar{z}}\right)\mathrm{I.P.}\left(w_{2zz} + 2w_{2z\bar{z}} + w_{2\bar{z}\bar{z}}\right)$$

$$+ \frac{1}{2}\mathrm{I.P.}\left(w_{1zz} - \bar{w}_{1\bar{z}\bar{z}}\right)\left(w_{2z} - w_{2\bar{z}}\right)$$

$$\psi_{1x}\psi_{1y}\psi_{1xy} = \frac{1}{4}\mathrm{I.P.}\, w_{1zz}\left(w_{1z}{}^2 - \bar{w}_{1\bar{z}}{}^2\right)$$

By means of these identities and by substituting for $w_{2z\bar{z}}$ from equation (29) of reference 1, equation (16) can be written as

$$\frac{8}{\mu^2 - 1} w_{3z\bar{z}} = -(\sigma + 2)\left(w_{2z} - w_{2\bar{z}}\right)\left(w_{1zz} - \bar{w}_{1\bar{z}\bar{z}}\right) + 2\left(w_{2z} + w_{2\bar{z}}\right)\left(w_{1zz} + \bar{w}_{1\bar{z}\bar{z}}\right)$$

$$- (\sigma + 2)\left(w_{2zz} + w_{2\bar{z}\bar{z}}\right)\left(w_{1z} + \bar{w}_{1\bar{z}}\right) + 2\left(w_{2zz} - w_{2\bar{z}\bar{z}}\right)\left(w_{1z} - \bar{w}_{1\bar{z}}\right)$$

$$+ \frac{1}{4}\sigma\left[\sigma\left(\mu^2 + 1\right) + (\sigma + 3)\left(\mu^2 - 1\right)\right] w_{1z}{}^2 w_{1zz}$$

$$+ \frac{1}{4}\left\{8(\sigma + 1) + \sigma^2\left(\mu^2 + 1\right) + \left(\mu^2 - 1\right)\left[\sigma + (\sigma + 2)(\sigma + 4)\right]\right\} \bar{w}_{1\bar{z}}{}^2 w_{1zz}$$

$$+ \frac{1}{2}\left[(\sigma + 2)^2\left(\mu^2 - 1\right) + \sigma(\sigma + 3)\left(\mu^2 + 1\right)\right] w_{1z}\bar{w}_{1\bar{z}} w_{1zz} \tag{17}$$

where ψ_3 is the imaginary part of w_3. From equation (33) of reference 1

$$w_2 = -\frac{1}{8}\left(\mu^2 - 1\right)\left[\frac{1}{2}\sigma\bar{z}w_{1z}{}^2 + (\sigma + 4)\bar{w}_1 w_{1z} + F(z)\right]$$

where $F(z)$ is a function, the form of which is decided by the boundary conditions but need not be given explicitly at this point. When the expressions for the derivatives of w_2 are inserted into equation (17), then

$$\frac{8}{\mu^2 - 1}w_{3z\bar{z}} = \frac{\sigma^2}{8}\left(\mu^2 - 1\right)\bar{z}\left(w_{1z}{}^2 w_{1zz}\right)_z + \frac{\sigma}{8}\left(\mu^2 - 1\right)\left(w_{1z}F_z\right)_z + \frac{\sigma(\sigma + 4)}{8}\left(\mu^2 - 1\right)\bar{w}_1\left(w_{1z}w_{1zz}\right)_z$$

$$+ \frac{\sigma(\sigma + 4)}{8}\left(\mu^2 - 1\right)w_{1z}\left(\bar{w}_{1\bar{z}}{}^2\right)_{\bar{z}} - \frac{\sigma + 4}{8}\left(\mu^2 - 1\right)\bar{w}_{1\bar{z}\bar{z}}F_z + \frac{\sigma + 4}{8}\left(\mu^2 - 1\right)\bar{w}_{1\bar{z}}F_{zz}$$

$$+ \frac{\sigma(\sigma + 4)}{8}\left(\mu^2 - 1\right)\bar{z}\bar{w}_{1\bar{z}}\left(w_{1z}w_{1zz}\right)_z + \frac{(\sigma + 4)^2}{16}\left(\mu^2 - 1\right)\left(\bar{w}_1{}^2\right)_{\bar{z}}w_{1zzz}$$

$$+ \left[\frac{\sigma^2}{16} + \frac{(\sigma + 4)^2}{8}\right]\left(\mu^2 - 1\right)w_{1z}{}^2\bar{w}_{1\bar{z}\bar{z}} - \frac{\sigma(\sigma + 4)}{16}\left(\mu^2 - 1\right)\left(\bar{z}\bar{w}_{1\bar{z}}\right)_{\bar{z}}\left(w_{1z}{}^2\right)_z + \frac{\sigma(\sigma + 3)}{2}\mu^2\bar{w}_{1\bar{z}}\left(w_{1z}{}^2\right)_z$$

$$- \frac{(\sigma + 4)^2}{8}\left(\mu^2 - 1\right)\left(\bar{w}_1\bar{w}_{1\bar{z}}\right)_{\bar{z}}w_{1zz} + \frac{1}{4}\left\{8(\sigma + 1) + \sigma^2\left(\mu^2 + 1\right) + \left[\sigma + (\sigma + 4)\left(\frac{3}{2}\sigma + 4\right)\right]\left(\mu^2 - 1\right)\right\}\bar{w}_{1\bar{z}}{}^2 w_{1zz}$$

$$+ \frac{\sigma}{12}\left[\frac{1}{2}\left(\frac{3}{2}\sigma + 4\right)\left(\mu^2 - 1\right) + \sigma\left(\mu^2 + 1\right)\right]\left(w_{1z}{}^3\right)_z \qquad (18)$$

This differential equation can be integrated by inspection without difficulty. Thus the general solution is

$$
\begin{aligned}
\frac{8}{\mu^2-1}w_3 = {} & \frac{\sigma^2}{16}\left(\mu^2-1\right)\bar{z}^2 w_{1z}{}^2 w_{1zz} + \frac{\sigma}{8}\left(\mu^2-1\right)\bar{z}w_{1z}F_z + \frac{\sigma(\sigma+4)}{8}\left(\mu^2-1\right)\bar{z}\bar{w}_1 w_{1z}w_{1zz} + \frac{\sigma(\sigma+4)}{8}\left(\mu^2-1\right)w_1\bar{w}_{1\bar{z}}{}^2 \\
& - \frac{\sigma+4}{8}\left(\mu^2-1\right)\bar{w}_{1\bar{z}}F + \frac{\sigma+4}{8}\left(\mu^2-1\right)\bar{w}_1 F_z + \frac{(\sigma+4)^2}{16}\left(\mu^2-1\right)\bar{w}_1{}^2 w_{1zz} - \frac{\sigma(\sigma+4)}{16}\left(\mu^2-1\right)\bar{z}\bar{w}_{1\bar{z}}w_{1z}{}^2 \\
& + \frac{\sigma(\sigma+3)}{2}\mu^2\bar{w}_1 w_{1z}{}^2 - \frac{(\sigma+4)^2}{8}\left(\mu^2-1\right)\bar{w}_1\bar{w}_{1\bar{z}}w_{1z} + \frac{1}{12}\sigma\left[\frac{1}{2}\left(\frac{3}{2}\sigma+4\right)\left(\mu^2-1\right)\right. \\
& \left. + \sigma\left(\mu^2+1\right)\right]\bar{z}w_{1z}{}^3 - \frac{1}{16}\left\{8(\sigma+2)^2 + \left[\sigma^2+2(3\sigma+8)(\sigma+2)\right]\left(\mu^2-1\right)\right\}\bar{w}_{1\bar{z}}\int w_{1z}{}^2\,dz \\
& + G_1(z) + G_2(\bar{z})
\end{aligned}
\tag{19}
$$

where $G_1(z)$ and $G_2(\bar{z})$ are arbitrary analytic functions, respectively, of only z and $\bar{z}$ to be determined by the boundary conditions. The boundary conditions to be satisfied are that at the surface of the ellipse, $\eta = 0$,

$$
\psi_3 = 0 \tag{20a}
$$

and at infinity, $\eta = \infty$,

$$\frac{\partial\psi_3}{\partial x} = \frac{\partial\psi_3}{\partial y} = 0 \tag{20(b)}$$

and that the trailing end of the major axis be a stagnation point. It may be noted that the result represented by equation (19) is not restricted to an elliptic profile but is valid for an arbitrary solid boundary.

Again, as in reference 1, the most direct way to impose the boundary condition, $\psi_3 = \text{I.P.}w_3 = 0$ at the surface of the solid, is to utilize the "polar" variable ζ of the z'-plane. Thus, for the elliptic profile according to equation (39) of reference 1,

$$\left.\begin{aligned} w_1 &= c \cos(\zeta + i\lambda) - 2R \cos\zeta - 2R\alpha(\sin\zeta + \zeta) \\ w_{1z} &= w_{1\zeta}\frac{d\zeta}{dz} = 1 - \frac{2R}{c}\frac{\sin\zeta}{\sin(\zeta + i\lambda)} + \frac{2R}{c}\alpha\frac{\cos\zeta + 1}{\sin(\zeta + i\lambda)} \end{aligned}\right\} \tag{21}$$

where, from equation (6)

$$z = c \cos(\zeta + i\lambda)$$

Just as in reference 1, it is a simple matter to supply the functions of ζ needed to satisfy the boundary condition at the surface of the ellipse in the z-plane. For example, at the surface where $\eta = 0$, $\cos\,(\bar{\zeta} - i\lambda) = \cos\,(\zeta - i\lambda)$. By use of equations (21) equation (19) then becomes

$$
\begin{aligned}
\frac{8}{\mu^2 - 1} w_3 = {} & \frac{\sigma^2}{16}\left(\mu^2 - 1\right)c^2\left[\cos^2(\bar{\zeta} - i\lambda) - \cos^2(\zeta - i\lambda)\right] {w_{1z}}^2 w_{1zz} \\
& + \frac{\sigma}{8}\left(\mu^2 - 1\right)c\,\left[\cos\,(\bar{\zeta} - i\lambda) - \cos\,(\zeta - i\lambda)\right] w_{1z} F_z \\
& + \frac{\sigma(\sigma + 4)}{8}\left(\mu^2 - 1\right)c^2\,\Big\{\cos\,(\bar{\zeta} - i\lambda)\,\Big[\cos\,(\bar{\zeta} - i\lambda) - 2\frac{R}{c}\cos\bar{\zeta} \\
& - 2\frac{R}{c}\alpha\,(\sin\bar{\zeta} + \bar{\zeta})\Big] - \cos\,(\zeta - i\lambda)\,\Big[\cos\,(\zeta - i\lambda) - 2\frac{R}{c}\cos\zeta \\
& - 2\frac{R}{c}\alpha\,(\sin\zeta + \zeta)\Big]\Big\} w_{1z} w_{1zz} \\
& + \frac{\sigma c}{8}\left[4(\sigma + 3) + (3\sigma + 8)\left(\mu^2 - 1\right)\right]\Big[\cos\,(\bar{\zeta} - i\lambda) - \frac{2R}{c}\cos\bar{\zeta} \\
& - \frac{2R}{c}\alpha\,(\sin\bar{\zeta} + \bar{\zeta}) - \cos\,(\zeta - i\lambda) + \frac{2R}{c}\cos\zeta + \frac{2R}{c}\alpha\,(\sin\zeta + \zeta)\Big] {w_{1z}}^2 \\
& + \frac{\sigma + 4}{8}\left(\mu^2 - 1\right)\Big[\frac{2R}{c}\,\frac{\sin\bar{\zeta}}{\sin\,(\bar{\zeta} - i\lambda)} - \frac{2R}{c}\alpha\,\frac{\cos\bar{\zeta} + 1}{\sin\,(\bar{\zeta} - i\lambda)} \\
& - \frac{2R}{c}\,\frac{\sin\zeta}{\sin\,(\zeta - i\lambda)} + \frac{2R}{c}\alpha\,\frac{\cos\zeta + 1}{\sin\,(\zeta - i\lambda)}\Big] F
\end{aligned}
$$

(equation continued on next page)

$$+ \frac{\sigma+4}{8}\left(\mu^2-1\right)c\left[\cos(\bar{\zeta}-i\lambda)-\frac{2R}{c}\cos\bar{\zeta}-\frac{2R}{c}\alpha\left(\sin\bar{\zeta}+\bar{\zeta}\right)\right.$$

$$\left.-\cos(\zeta-i\lambda)+\frac{2R}{c}\cos\zeta+\frac{2R}{c}\alpha\left(\sin\zeta+\zeta\right)\right]F_z$$

$$+\frac{(\sigma+4)^2}{16}\left(\mu^2-1\right)c^2\left\{\left[\cos(\bar{\zeta}-i\lambda)-\frac{2R}{c}\cos\bar{\zeta}-\frac{2R}{c}\alpha\left(\sin\bar{\zeta}+\bar{\zeta}\right)\right]^2\right.$$

$$\left.-\left[\cos(\zeta-i\lambda)-\frac{2R}{c}\cos\zeta-\frac{2R}{c}\alpha\left(\sin\zeta+\zeta\right)\right]^2\right\}w_{1zz}$$

$$+\frac{\sigma(\sigma+4)}{16}\left(\mu^2-1\right)c\left\{\cos(\bar{\zeta}-i\lambda)\left[-1+\frac{2R}{c}\frac{\sin\bar{\zeta}}{\sin(\bar{\zeta}-i\lambda)}\right.\right.$$

$$\left.-\frac{2R}{c}\alpha\frac{\cos\bar{\zeta}+1}{\sin(\bar{\zeta}-i\lambda)}\right]-\cos(\zeta-i\lambda)\left[-1+\frac{2R}{c}\frac{\sin\zeta}{\sin(\zeta-i\lambda)}\right.$$

$$\left.\left.-\frac{2R}{c}\alpha\frac{\cos\zeta+1}{\sin(\zeta-i\lambda)}\right]\right\}w_{1z}{}^2+\frac{(\sigma+4)^2}{8}\left(\mu^2-1\right)\left[-\frac{2R}{c}\frac{\sin\bar{\zeta}}{\sin(\bar{\zeta}-i\lambda)}\right.$$

$$\left.+\frac{2R}{c}\alpha\frac{\cos\bar{\zeta}+1}{\sin(\bar{\zeta}-i\lambda)}+\frac{2R}{c}\frac{\sin\zeta}{\sin(\zeta-i\lambda)}-\frac{2R}{c}\alpha\frac{\cos\zeta+1}{\sin(\zeta-i\lambda)}\right]w_1w_{1z}$$

$$+\frac{\sigma c}{48}\left[8\sigma+(7\sigma+8)\left(\mu^2-1\right)\right]\left[\cos(\bar{\zeta}-i\lambda)-\cos(\zeta-i\lambda)\right]w_{1z}{}^3$$

$$+\frac{1}{16}\left\{8(\sigma+2)^2+\left[\sigma^2+2(\sigma+2)(3\sigma+8)\right]\left(\mu^2-1\right)\right\}\left[\frac{2R}{c}\frac{\sin\bar{\zeta}}{\sin(\bar{\zeta}-i\lambda)}\right.$$

$$-\frac{2R}{c}\alpha\frac{\cos\bar{\zeta}+1}{\sin(\bar{\zeta}-i\lambda)}-\frac{2R}{c}\frac{\sin\zeta}{\sin(\zeta-i\lambda)}$$

$$\left.+\frac{2R}{c}\alpha\frac{\cos\zeta+1}{\sin(\zeta-i\lambda)}\right]\int_{S.P.}^{\zeta}w_{1z}{}^2\frac{dz}{d\zeta}\,d\zeta \tag{22}$$

where

$$\int_{S.P.}^{\zeta} w_{1z}^{2} \frac{dz}{d\zeta} d\zeta = c \left\{ \cos(\zeta + i\lambda) - \frac{4R}{c}\alpha(\sin\zeta + \zeta) - \frac{4R}{c}\cos\zeta + \frac{2R^2}{c^2}\sinh^2\lambda \log\frac{\cos(\zeta + i\lambda) - 1}{\cos(\zeta + i\lambda) + 1} \right.$$

$$+ \frac{4R^2}{c^2}\cos(\zeta - i\lambda) + \frac{4R^2}{c^2}\alpha\left[2\sin(\zeta - i\lambda) + 2\zeta\cosh\lambda - \frac{1}{2}\sinh 2\lambda \log\frac{\cos(\zeta + i\lambda) - 1}{\cos(\zeta + i\lambda) + 1}\right.$$

$$\left.\left. - 2i\sinh\lambda \log\sin(\zeta + i\lambda)\right]\right\} + c\left\{\cosh\lambda + \frac{4R}{c}(\pi\alpha - 1) - \frac{2R^2}{c^2}\sinh^2\lambda \log\frac{\cosh\lambda + 1}{\cosh\lambda - 1}\right.$$

$$+ \frac{4R^2}{c^2}\cosh\lambda - \frac{4R^2}{c^2}\alpha\left[2i\sinh\lambda + 2\pi\cosh\lambda - \frac{1}{2}\sinh 2\lambda \log\frac{\cosh\lambda + 1}{\cosh\lambda - 1}\right.$$

$$\left.\left. - 2i\sinh\lambda \log(-i\sinh\lambda)\right]\right\}$$

and S.P., the lower limit of integration, denotes the stagnation point. By means of the following formula (equation (44) of reference 1) for the complex velocity (with regard to ψ_3 only) in the physical Z-plane:

$$\frac{\rho}{\rho_1 U}\left(u_3 - \frac{1}{\mu}v_3\right) = 2i\frac{d\zeta}{dz}\frac{\partial\psi_3}{\partial\zeta} = \frac{d\zeta}{dz}\frac{\partial}{\partial\zeta}\left(w_3 - \bar{w}_3\right) \tag{24}$$

equation (22) can be shown to fulfill the boundary conditions, not only at the solid surface, $\eta = 0$, but also at infinity, $\eta = \infty$; moreover, the trailing end of the major axis $(\xi = \pi, \eta = 0)$ is a stagnation point. In order to satisfy the boundary conditions, however, a number of singularities of the nature of doublets have been introduced into the field of flow. These unwelcome singularities are caused by the factor $1/\sin(\zeta - i\lambda)$ and are located at the exterior points $z' = \pm Re^{\lambda}$ or $\zeta_1 = i\lambda$ and $\zeta_2 = \pi + i\lambda$. They are removed by the addition of doublets in such a manner that the sum of the residues at a pole is zero. The images, moreover, of these superimposed singularities in the conformal circle of radius R must be included in order to insure that the boundary conditions are preserved. As an example, consider the expression

$$\frac{H(\zeta)}{\sin(\zeta - i\lambda)} \tag{25}$$

where the function $H(\zeta)$ is regular everywhere in the finite region exterior to the circle of radius R $(\eta = 0)$. Then, in order to cancel the residues at the poles $\zeta_1 = i\lambda$ and $\zeta_2 = \pi + i\lambda$ and to preserve the boundary conditions at the solid and at infinity, the following expression must be added to the right-hand side of equation (22):

$$-\frac{1}{2}H(i\lambda)\left(\cot\frac{\zeta - i\lambda}{2} - \cot\frac{\bar{\zeta} - i\lambda}{2}\right) - \frac{1}{2}H(\pi + i\lambda)\left(\tan\frac{\zeta - i\lambda}{2} - \tan\frac{\bar{\zeta} - i\lambda}{2}\right) \tag{26}$$

By means of this expression the additional terms can be easily obtained in order that w_3 be regular everywhere in the finite region exterior to the conformal circle of radius R in the z'-plane or to the elliptic profiles in the z-plane and Z-plane. An examination of equation (22) immediately yields the following equation for $H(\zeta)$:

$$H(\zeta) = \frac{\sigma+4}{8}\left(\mu^2 - 1\right)e^{\lambda}\left[-\sin\zeta + \alpha\,(\cos\zeta + 1)\right]\left[F(\zeta) + \frac{\sigma c}{2}\cos\,(\zeta - i\lambda)w_{1z}{}^2\right.$$

$$\left. - (\sigma+4)w_1 w_{1z}\right] + \frac{1}{16}\left\{8(\sigma+2)^2 + \left[\sigma^2 + 2(\sigma+2)(3\sigma+8)\right]\left(\mu^2 - 1\right)\right\}\left[-\sin\zeta\right.$$

$$\left. + \alpha\,(\cos\zeta + 1)\right]e^{\lambda}\int_{S.P.}^{\zeta} w_{1z}{}^2\,\frac{dz}{d\zeta}\,d\zeta \tag{27}$$

where, from equation (43) of reference 1,

$$F(\zeta) = -\frac{1}{2}\sigma c\,\cos\,(\zeta - i\lambda)w_{1z}{}^2 - 2(\sigma+4)ic\,\sinh\lambda\,\sin\zeta w_{1z} - (\sigma+4)w_1 w_{1z}$$

$$+ 2b(\sigma+4)\alpha\,(\sin\zeta + \zeta) \tag{28}$$

Now, the introduction of the foregoing singularities induces a finite velocity at the trailing end of the major axis of the elliptic boundary. This velocity, obtained by means of equation (24) and expression (26), is

$$\frac{\rho}{\rho_1 U}\left(u_3 - \frac{i}{\mu}v_3\right)_{S.P.} = \frac{1}{4ic\,\sinh\lambda}\left\{\left[H(i\lambda) + \overline{H}(-i\lambda)\right]\frac{1}{\cosh^2\frac{\lambda}{2}}\right.$$

$$\left. + \left[H(\pi + i\lambda) + \overline{H}(\pi - i\lambda)\right]\frac{1}{\sinh^2\frac{\lambda}{2}}\right\} \tag{29}$$

Then, by means of equations (27) and (28), with terms of higher order than the third neglected

$$\frac{\rho}{\rho_1 U}\left(u_3 - \frac{i}{\mu}v_3\right)_{S.P.} = \frac{i\alpha}{24}(\sigma + 4)^2\left(\mu^2 - 1\right)^2\frac{b}{a + b}$$

$$- \frac{i\alpha}{128}(3 - \log 4)\left(\mu^2 - 1\right)\left\{8(\sigma + 2)^2\right.$$

$$\left. + \left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\left(\mu^2 - 1\right)\right\}\frac{b}{a + b}$$

In order to maintain $(\xi = \pi, \eta = 0)$ a stagnation point, the following expression, the imaginary part of which vanishes for $\eta = 0$ and for $\eta = \infty$,

$$\left(\frac{1}{3}(\sigma + 4)^2\left(\mu^2 - 1\right) - \frac{1}{16}(3 - \log 4)\left\{8(\sigma + 2)^2\right.\right.$$

$$\left.\left. + \left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\left(\mu^2 - 1\right)\right\}\right)\frac{b^2}{a + b}\alpha\zeta \qquad (30)$$

must be added to the right-hand side of equation (22). Finally then, the complete expression for $\frac{8}{\mu^2 - 1}w_3$ is given by the right-hand side of equation (22) and the expressions (26) and (30).

EQUATIONS FOR $\left(\frac{\rho q}{\rho_1 U}\right)^2$ AND ρ_1/ρ

The components u_3 and v_3 of the velocity of the compressible fluid in the physical flow plane Z are calculated by means of equation (24). Since, for the purpose of the present paper, calculations are performed along an ultimately infinitely large contour, the developments for u_3 and v_3 in the neighborhood of infinity are sufficient. Thus, by use of the complete expression for w_3, the development of the complex velocity $\frac{\rho}{\rho_1 U}\left(u_3 - \frac{i}{\mu}v_3\right)$ in the neighborhood of infinity is given by

$$\frac{\rho}{\rho_1 U}\left(u_3 - \frac{i}{\mu}v_3\right) = -i\alpha A\left(\frac{b}{a+b}\right)^2 \frac{1}{z' e^{\eta}} + i\alpha\left(B_1 + B_2 \log \frac{b}{a}\right)\left(\frac{b}{a+b}\right)^2 \frac{1}{z'^2 e^{2\eta}}$$

$$- i\alpha C\left(\frac{b}{a+b}\right)^2 \frac{z'^2}{e^{2\eta}} + i\alpha D\left(\frac{b}{a+b}\right)^2 \frac{1}{z'^4 e^{2\eta}} + E\left[\mu\alpha\left(\frac{b}{a+b}\right)^2\right.$$

$$\left. - 6\left(\frac{b}{a+b}\right)^3\right] \frac{1}{z'^2 e^{2\eta}} + \ldots \tag{31}$$

where

$$z' = e^{-i\xi}$$

and

$$A = \frac{1}{24}(\sigma + 4)^2\left(\mu^2 - 1\right)^2 + \frac{1}{64}(3 - \log 4)\left(\mu^2 - 1\right)\left\{8(\sigma + 2)^2\right.$$

$$\left. + \left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\left(\mu^2 - 1\right)\right\}$$

$$B_1 = \frac{1}{16}\left(\mu^2 - 1\right)\left[8\left(2\sigma^2 + 7\sigma + 4\right) + \left(\mu^2 - 1\right)\left(13\sigma^2 + 44\sigma + 32\right)\right]$$

$$B_2 = \frac{1}{16}\left(\mu^2 - 1\right)\left\{8(\sigma + 2)^2 + \left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\left(\mu^2 - 1\right)\right\}$$

$$C = \frac{1}{16}(\sigma + 4)(\sigma + 8)\left(\mu^2 - 1\right)^2$$

$$D = \frac{1}{2}(\sigma + 4)\left(\mu^2 - 1\right)^2$$

$$E = \frac{1}{16}(\sigma + 4)^2\left(\mu^2 - 1\right)^2$$

From equation (31), it follows easily that

$$\left.\begin{aligned}
2\frac{\rho u_3}{\rho_1 U} = {} & i\alpha A\left(\frac{b}{a+b}\right)^2\left(z' - \frac{1}{z'}\right)\frac{1}{e^{\eta}} - i\alpha\left(B_1 + B_2 \log\frac{b}{a} + C\right)\left(\frac{b}{a+b}\right)^2\left(z'^2 - \frac{1}{z'^2}\right)\frac{1}{e^{2\eta}} \\
& - i\alpha D\left(\frac{b}{a+b}\right)^2\left(z'^4 - \frac{1}{z'^4}\right)\frac{1}{e^{2\eta}} \\
& + E\left(\pi\alpha - 6\frac{b}{a+b}\right)\left(\frac{b}{a+b}\right)^2\left(z'^2 + \frac{1}{z'^2}\right)\frac{1}{e^{2\eta}} + \cdots \\
2\frac{1}{\mu}\frac{\rho v_3}{\rho_1 U} = {} & i\alpha A\left(\frac{b}{a+b}\right)^2\left(z' + \frac{1}{z'}\right)\frac{1}{e^{\eta}} \\
& - i\alpha\left(B_1 + B_2 \log\frac{b}{a} - C\right)\left(\frac{b}{a+b}\right)^2\left(z'^2 + \frac{1}{z'^2}\right)\frac{1}{e^{2\eta}} \\
& - i\alpha D\left(\frac{b}{a+b}\right)^2\left(z'^4 + \frac{1}{z'^4}\right)\frac{1}{e^{2\eta}} \\
& + E\left(\pi\alpha - 6\frac{b}{a+b}\right)\left(\frac{b}{a+b}\right)^2\left(z'^2 - \frac{1}{z'^2}\right)\frac{1}{e^{2\eta}} + \cdots
\end{aligned}\right\} \quad (32)$$

Then

$$\frac{\rho}{\rho_1 U}\left(u_3 - iv_3\right) = \frac{i\alpha}{2}A\left(\frac{b}{a+b}\right)^2\left[(1-\mu)z' - (1+\mu)\frac{1}{z'}\right]\frac{1}{e^{\eta}}$$

$$- \frac{i\alpha}{2}\left(B_1 + B_2 \log\frac{b}{a}\right)\left(\frac{b}{a+b}\right)^2\left[(1-\mu)z'^2 - (1+\mu)\frac{1}{z'^2}\right]\frac{1}{e^{2\eta}}$$

$$- \frac{i\alpha}{2}C\left(\frac{b}{a+b}\right)^2\left[(1+\mu)z'^2 - (1-\mu)\frac{1}{z'^2}\right]\frac{1}{e^{2\eta}}$$

$$- \frac{i\alpha}{2}D\left(\frac{b}{a+b}\right)^2\left[(1-\mu)z'^4 - (1+\mu)\frac{1}{z'^4}\right]\frac{1}{e^{2\eta}}$$

$$+ \frac{1}{2}E\left(\pi\alpha - 6\frac{b}{a+b}\right)\left(\frac{b}{a+b}\right)^2\left[(1-\mu)z'^2 + (1+\mu)\frac{1}{z'^2}\right]\frac{1}{e^{2\eta}} + \ldots \tag{33}$$

From equation (46) of reference 1 and equation (33), therefore, the development in the neighborhood of infinity of the complete complex velocity, inclusive of terms of the third order, is given by

$$\frac{\rho}{\rho_1 U}(u - iv) = -1 - i\mu\alpha + \frac{i\alpha}{e^{\eta}}\left[1 + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\frac{b}{a+b} + \frac{1}{2}A\left(\frac{b}{a+b}\right)^2\right]\left[(1-\mu)z'\right.$$

$$\left. - (1+\mu)\frac{1}{z'}\right] + \frac{i\alpha}{e^{2\eta}}\left\{\frac{a}{a+b} - \left[\left(\mu^2 - 1\right) + \frac{1}{2}B_1 + \frac{1}{2}B_2 \log\frac{b}{a}\right.\right.$$

$$\left.\left. + \frac{1}{2}C\right]\left(\frac{b}{a+b}\right)^2\right\}\left[(1-\mu)z'^2 - (1+\mu)\frac{1}{z'^2}\right] - \frac{i\alpha}{e^{2\eta}}\mu\left[\left(\mu^2 - 1\right)\frac{b}{a+b}\right.$$

$$\left. + C\left(\frac{b}{a+b}\right)^2\right]\left(z'^2 + \frac{1}{z'^2}\right) - \frac{i\alpha}{e^{2\eta}}\left[\left(\mu^2 - 1\right)\frac{b}{a+b}\right.$$

$$\left. + \frac{1}{2}D\left(\frac{b}{a+b}\right)^2\right]\left[(1-\mu)z'^4 - (1+\mu)\frac{1}{z'^4}\right] + \frac{1}{e^{2\eta}}\left\{\frac{b}{a+b}\right.$$

$$+ \left[\frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\right.$$

$$\left.\left. + \frac{1}{2}E\left(\pi\alpha - 6\frac{b}{a+b}\right)\right]\left(\frac{b}{a+b}\right)^2\right\}\left[(1-\mu)z'^2 + (1+\mu)\frac{1}{z'^2}\right] + \ldots \tag{34}$$

From equation (34),

$$\left(\frac{\rho q}{\rho_1 U}\right)^2 = \left(\frac{\rho}{\rho_1 U}\right)^2 (u - iv)(u + iv)$$

$$= 1 - \frac{2i\alpha}{e^{\eta}}\left[1 + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\frac{b}{a + b} + \frac{1}{2}A\left(\frac{b}{a + b}\right)^2\right]\left(z' - \frac{1}{z'}\right) - \frac{2i\alpha}{e^{2\eta}}\left\{\frac{a}{a + b}\right.$$

$$\left. - \left[\left(\mu^2 - 1\right) + \frac{1}{2}B_1 + \frac{1}{2}B_2 \log\frac{b}{a} + \frac{1}{2}C\right]\left(\frac{b}{a + b}\right)^2\right\}\left(z'^2 - \frac{1}{z'^2}\right)$$

$$+ \frac{2i\alpha}{e^{2\eta}}\left[\left(\mu^2 - 1\right)\frac{b}{a + b} + \frac{1}{2}D\left(\frac{b}{a + b}\right)^2\right]\left(z'^4 - \frac{1}{z'^4}\right) - \frac{2i\alpha}{e^{2\eta}}\mu^2\left[\frac{b}{a + b}\right.$$

$$\left. + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\left(\frac{b}{a + b}\right)^2\right]\left(z'^2 - \frac{1}{z'^2}\right) - \frac{2}{e^{2\eta}}\left\{\frac{b}{a + b} + \left[\frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\right.\right.$$

$$\left.\left. + \frac{1}{2}E\left(\pi\alpha - 6\frac{b}{a + b}\right)\right]\left(\frac{b}{a + b}\right)^2\right\}\left(z'^2 + \frac{1}{z'^2}\right) + \ldots \tag{35}$$

Now, by definition

$$\frac{\rho u}{\rho_1 U} = \frac{1}{U}\psi_Y$$

and

$$\frac{\rho v}{\rho_1 U} = -\frac{1}{U}\psi_X$$

Hence

$$\frac{\psi_X{}^2 + \psi_Y{}^2}{U^2} = \left(\frac{\rho q}{\rho_1 U}\right)^2$$

and by means of equation (17) of reference 1 for $\frac{\rho_1}{\rho}$ and equation (35),

$$\begin{aligned}
\frac{\rho_1}{\rho} = 1 &- \frac{i\alpha}{e^{\eta}}\left(\mu^2 - 1\right)\left[1 + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\frac{b}{a+b} + \frac{1}{2}A\left(\frac{b}{a+b}\right)^2\right]\left(z' - \frac{1}{z'}\right) \\
&- \frac{i\alpha}{e^{2\eta}}\left(\mu^2 - 1\right)\left\{\frac{a}{a+b} - \left[\left(\mu^2 - 1\right) + \frac{1}{2}B_1 + \frac{1}{2}B_2 \log\frac{b}{a}\right.\right. \\
&\left.\left. + \frac{1}{2}C\right]\left(\frac{b}{a+b}\right)^2\right\}\left(z'^2 - \frac{1}{z'^2}\right) + \frac{i\alpha}{e^{2\eta}}\left(\mu^2 - 1\right)\left[\left(\mu^2 - 1\right)\frac{b}{a+b}\right. \\
&\left. + \frac{1}{2}D\left(\frac{b}{a+b}\right)^2\right]\left(z'^4 - \frac{1}{z'^4}\right) - \frac{i\alpha}{e^{2\eta}}\mu^2\left(\mu^2 - 1\right)\left[\frac{b}{a+b}\right. \\
&\left. + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\left(\frac{b}{a+b}\right)^2\right]\left(z'^2 - \frac{1}{z'^2}\right) - \frac{1}{e^{2\eta}}\left(\mu^2 - 1\right)\left\{\frac{b}{a+b}\right. \\
&\left. + \left[\frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4) + \frac{1}{2}E\left(\pi\alpha - 6\frac{b}{a+b}\right)\right]\left(\frac{b}{a+b}\right)^2\right\}\left(z'^2 + \frac{1}{z'^2}\right) + \ldots \qquad (36)
\end{aligned}$$

CALCULATION OF THE LIFT

In a compressible flow as in an incompressible flow, the lift is given by

$$L_c = \rho_1 U \Gamma_c$$

where Γ_c is the circulation round the profile and where, by definition,

$$\Gamma_c = \oint (u\,dX + v\,dY) = R.P.\oint (u - iv)\,dZ$$

Now, dZ is given by equation (7b), and from equations (34) and (36)

$$\frac{1}{U}(u - iv) = -1 - i\mu\alpha + \frac{i\alpha}{e^{\eta}}\left[1 + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\frac{b}{a + b}\right.$$

$$\left.+ \frac{1}{2}A\left(\frac{b}{a + b}\right)^2\right]\left[(1 - \mu)z' - (1 + \mu)\frac{1}{z'} + \left(\mu^2 - 1\right)\left(z' - \frac{1}{z'}\right)\right] + \ldots$$

Then, because only terms that involve dz'/z' contribute to the line integral, it follows that

$$\Gamma_c = 4\pi RU\mu^2\alpha\left[1 + \frac{1}{8}\left(\mu^2 - 1\right)(\sigma + 4)\frac{b}{R} + \frac{1}{8}A\left(\frac{b}{R}\right)^2\right]$$

or if A is replaced by its definition (see equation (31)), it follows that

$$\Gamma_c = 4\pi RU\mu^2\alpha\left[1 + \frac{1}{8}\left(\mu^2 - 1\right)(\sigma + 4)\frac{b}{R} + \frac{1}{64}\left(\frac{1}{3}(\sigma + 4)^2\left(\mu^2 - 1\right)^2\right.\right.$$

$$\left.\left. + \frac{1}{8}(3 - \log 4)\left(\mu^2 - 1\right)\left\{8(\sigma + 2)^2 + \left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\left(\mu^2 - 1\right)\right\}\right)\frac{b^2}{R^2}\right] \tag{37}$$

If, according to the correspondence equations (9), b, α, and R are replaced by $\frac{1}{\mu}b'$, $\frac{1}{\mu}\alpha'$, and $R' + \frac{1 - \mu}{2\mu}b'$, respectively, then for the actual ellipse in the physical flow plane,

$$\Gamma_c = 4\pi R'U\alpha'\mu + 2\pi U\alpha'b'\left[(1-\mu) + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4) + \frac{1}{32}\left(\frac{1}{3}(\sigma + 4)^2\left(\mu^2 - 1\right)^2\right.\right.$$

$$\left.\left. + \frac{1}{8}(3 - \log 4)\left(\mu^2 - 1\right)\left\{8(\sigma + 2)^2 + \left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\left(\mu^2 - 1\right)\right\}\right)\frac{b'}{\mu R'}\right]$$

Since the circulation in the case of incompressible flow is

$$\Gamma_i = 4\pi R'U\alpha'$$

the ratio Γ_c/Γ_i or L_c/L_i is given by

$$\frac{L_c}{L_i} = \frac{\Gamma_c}{\Gamma_i} = \mu + \frac{t'}{1 + t'}\left[\mu(\mu - 1) + \frac{1}{4}(\gamma + 1)\left(\mu^2 - 1\right)^2\right]$$

$$+ \frac{1}{16}\,\frac{\mu^2 - 1}{\mu}\left(\frac{t'}{1 + t'}\right)^2\left(\frac{1}{3}\left(\mu^2 - 1\right)(\sigma + 4)^2 + \frac{1}{8}(3 - \log 4)\left\{8(\sigma + 2)^2\right.\right.$$

$$\left.\left. + \left(\mu^2 - 1\right)\left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\right\}\right) \tag{38}$$

where t' is the thickness coefficient b'/a' of the actual elliptic profile in the physical flow plane. Equation (38) represents a second-step improvement of the Prandtl-Glauert approximation and reduces to that result when $t' \longrightarrow 0$. In reference 1 a first-step improvement of the Prandtl-Glauert approximation for the ratio L_c/L_i was calculated and is represented by the first two terms on the right-hand side of equation (38). Table I shows values of the ratio L_c/L_i for the first-step and second-step improvements, for various values of the thickness coefficient t' and the stream Mach number M_1 (with $\gamma = 1.4$ for air). Figure 2 shows the corresponding graphs with M_1 as abscissa and L_c/L_i as ordinate. An examination of these graphs shows that below the critical stream Mach number M_{cr} the main effect of compressibility is already given by the Prandtl-Glauert term and the first-step improvement and that large differences between the first-step and second-step improvements do not appear until well above the critical stream Mach number.

In reference 2 the ratio L_c/L_i was calculated for an elliptic cylinder by the method of Poggi. This result, restated in the notation of the present paper, is

$$\frac{L_c}{L_i} = 1 + \frac{1}{2}M_1^{\,2}\left[\frac{1 + 2t'}{1 - t'} + 2\frac{1 + t'}{1 - t'}\log\frac{2}{1 + t'} + \frac{2 - t'^2}{(1 - t')^2}\log\frac{1}{t'} - \sqrt{\frac{1 + t'}{1 - t'}}\,\frac{2}{1 - t'}\log\frac{\sqrt{1 + t'} + \sqrt{1 - t'}}{\sqrt{1 + t'} - \sqrt{1 - t'}}\right] + \ldots \tag{39}$$

and must agree with equation (38) insofar as the terms common to the two developments are concerned. If, then, equation (38) is expanded according to powers of $M_1^{\,2}$ and equation (39) is expanded according to powers of t', the two expansions are found to agree and yield

$$\frac{L_c}{L_i} = 1 + \frac{1}{2}M_1^{\,2} + \frac{1}{2}M_1^{\,2}t' + \frac{1}{4}(1 - \log 4)M_1^{\,2}t'^2 + \ldots$$

CALCULATION OF THE MOMENT

For the purpose of calculating the moment, the following two equations are needed. From equations (7)

$$Z\,dZ = -\frac{c^2e^{2\eta+2\lambda}}{16}\left[(1+\mu)^2\left(\frac{1}{z'^2e^{4\eta+4\lambda}} - z'^2\right) + (1-\mu)^2\left(\frac{1}{z'^2} - \frac{z'^2}{e^{4\eta+4\lambda}}\right)\right]\frac{dz'}{z'} - \frac{c^2}{8}\left(1-\mu^2\right)\left(\frac{1}{z'^2} - z'^2\right)\frac{dz'}{z'}$$

and from equations (34) and (36)

$$\frac{\rho_1}{\rho}\left[\frac{\rho}{\rho_1 U}(u - iv)\right]^2 = -\frac{i\alpha}{e^{2\eta}}\left\{\frac{a}{a+b} - \left[\left(\mu^2 - 1\right) + \frac{1}{2}B_1 + \frac{1}{2}B_2 \log\frac{b}{a}\right.\right.$$

$$\left.\left. + \frac{1}{2}C\right]\left(\frac{b}{a+b}\right)^2\right\}\left[(1-\mu)^2 z'^2 - (1+\mu)^2\frac{1}{z'^2}\right]$$

$$+ \frac{2i\alpha}{e^{2\eta}}\mu\left[\left(\mu^2 - 1\right)\frac{b}{a+b} + C\left(\frac{b}{a+b}\right)^2\right]\left(z'^2 + \frac{1}{z'^2}\right)$$

$$- \frac{i\alpha}{e^{2\eta}}\mu\left(\mu^2 - 1\right)\left[\frac{b}{a+b} + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\left(\frac{b}{a+b}\right)^2\right]\left[(1+\mu)z'^2\right.$$

$$\left. + (1-\mu)\frac{1}{z'^2}\right] - \frac{i\alpha}{e^{2\eta}}\mu\left[\frac{b}{a+b}\right.$$

$$\left. + \frac{1}{4}\left(\mu^2 - 1\right)(\sigma + 4)\left(\frac{b}{a+b}\right)^2\right]\left[(1-\mu)^2 z'^2\right.$$

$$\left. + (1+\mu)^2\frac{1}{z'^2}\right] + \ldots$$

By means of these equations and equations (8), (35), and (36), equation (5) for the moment M_c about the origin yields the following result:

$$M_c = \pi\rho_1 U^2\alpha\frac{(a+b)^2}{8}\left(8\mu^2\frac{a-b}{a+b} - \mu^2\left\{\left(\mu^2 - 1\right)\left[8 + \left(\mu^2 + 1\right)(\sigma + 4)\right]\right.\right.$$

$$\left.\left. + 4B_1 + 4B_2 \log\frac{b}{a}\right\}\left(\frac{b}{a+b}\right)^2\right)$$

Again, replacing a, b, c, and α according to the correspondence equations (9) yields

$$M_c = \pi\rho_1 U^2\alpha' c'^2\mu - \frac{1}{8}\pi\rho_1 U^2\alpha'\frac{\mu^4 - 1}{\mu}(\sigma + 4)b'^2 - \frac{1}{32}\pi\rho_1 U^2\alpha'\frac{\mu^2 - 1}{\mu}\left(\left\{8(\sigma + 2)^2 + \left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\left(\mu^2 - 1\right)\right\}\log\frac{b'}{\mu a'} + \left[8\left(2\sigma^2 + 7\sigma + 4\right) + \left(13\sigma^2 + 44\sigma + 32\right)\left(\mu^2 - 1\right)\right]\right)b'^2$$

where B_1 and B_2 have been replaced by their definitions. (See equation 31.) Now, for an incompressible fluid,

$$M_i = \pi\rho_1 U^2\alpha' c'^2$$

The ratio M_c/M_i for the actual elliptic profile in the physical flow plane therefore becomes

$$\frac{M_c}{M_i} = \mu - \frac{1}{32}\,\frac{\mu^2 - 1}{\mu}\left(16(\sigma + 2)^2 + \left(\mu^2 - 1\right)\left[\sigma^2 + 12(\sigma + 2)^2\right] - \left\{8(\sigma + 2)^2 + \left(\mu^2 - 1\right)\left[\sigma^2 + 2(\sigma + 2)(3\sigma + 8)\right]\right\}\log\frac{\mu}{t'}\right)\frac{t'^2}{1 - t'^2} \tag{40}$$

Equation (40) represents the complete first-step improvement of the Prandtl-Glauert approximation for the ratio of moments M_c/M_i and reduces to that result in the limiting case $t' \longrightarrow 0$. Again, as in the case of the lift, the ratio M_c/M_i was calculated for an elliptic cylinder by the method of Poggi (reference 2). This result, restated in the notation of the present paper, is

$$\frac{M_c}{M_i} = 1 + \frac{1}{2}M_1^2 + \frac{t'^2}{(1 - t')^2}\left(\frac{1 + t'}{1 - t'}\log\frac{1}{t'} - 2\right)M_1^2 + \ldots \tag{41}$$

Just as in the case of the lift, equations (40) and (41) must agree insofar as the terms common to the two results are concerned. Thus, if equation (40) is expanded according to powers of M_1^2 and equation (41) is expanded according to powers of t', the expansions are found to agree and yield

$$\frac{M_c}{M_i} = 1 + \frac{1}{2}M_1^2 - 2M_1^2t'^2 + M_1^2t'^2 \log \frac{1}{t'} + \cdot \cdot \cdot$$

Table II shows values of the ratio M_c/M_i calculated by means of equation (40) for various values of the thickness coefficient t' and the stream Mach number M_1, and figure 2 shows the corresponding graphs with M_1 as abscissa and M_c/M_i as ordinate.

Contrary to the lift, which is a localized vector, the moment is a nonlocalized vector, the magnitude of which depends on the point about which it is taken. In the present paper, this point is the origin of coordinates. If, now, the moment about the origin O is denoted by M_{c_O}, the moment about any other point P in the plane of flow is then given by

$$M_{c_P} = M_{c_O} - rL_c$$

where r is the length of the perpendicular dropped from the origin O to the line of action of the lift vector L_c through the point P. If this expression for the moment is examined in relation to the moment about the same point P in an incompressible fluid, it will be seen that the ratio of moments M_{c_P}/M_{i_P} again begins with the Prandtl-Glauert approximation but that the higher terms of the approximation depend on the point P about which the moments are taken. Figure 2, consequently, should not be used to compare the various moment curves with the Prandtl-Glauert approximation; rather, the significant result is the compressibility effect on the movement of the center of pressure from its position in an incompressible fluid - a quantity that is independent of the point about which moments are taken.

EFFECT OF COMPRESSIBILITY ON POSITION OF CENTER OF PRESSURE

If C_c and C_i denote the distance of the center of pressure from the center of the ellipse in the compressible and incompressible fluids, respectively, then

$$\frac{C_c}{C_i} = \frac{M_c}{M_i}\frac{L_i}{L_c}$$

Also, $C_i = \frac{a}{2}(1 - t')$; therefore,

$$\frac{C_c - C_i}{2a} = \frac{1 - t'}{4}\left(\frac{M_c}{M_i}\frac{L_i}{L_c} - 1\right) \tag{42}$$

By means of this formula and equations (38) and (40) it is possible to calculate the effect of compressibility on the position of the center of pressure for various thickness coefficients and stream Mach numbers. Table III shows values of the ratio $\frac{C_c - C_i}{2a}$, the negative values indicating movement toward the center of the elliptic profile. Figure 2 shows the corresponding graphs with the stream Mach number M_i as abscissa and the ratio $\frac{C_c - C_i}{2a}$ in percent chord as ordinate. Note that in each case at some high subsonic stream Mach number, the movement of the center of pressure reverses. (See table III where sign changes from negative to positive.) This peculiar behavior of the center of pressure is probably caused by the term $\log \frac{\mu}{t'}$ in the equation for the moment M_c and indicates the need for additional terms in the expansion for the stream function ψ to insure greater accuracy in the range of high subsonic stream Mach numbers.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., October 24, 1946

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1. Kaplan, Carl: Effect of Compressibility at High Subsonic Velocities on the Lifting Force Acting on an Elliptic Cylinder. NACA TN No. 1118, 1946.

2. Kaplan, Carl: A Theoretical Study of the Moment on a Body in a Compressible Fluid. NACA Rep. No. 671, 1939.

TABLE I

RATIO OF LIFTS FOR COMPRESSIBLE AND INCOMPRESSIBLE FLOWS

M_1	μ	L_c/L_i (first-step improvement)				L_c/L_i (second-step improvement)			
		$t' = 0.05$	$t' = 0.10$	$t' = 0.15$	$t' = 0.20$	$t' = 0.05$	$t' = 0.10$	$t' = 0.15$	$t' = 0.20$
0.10	1.0050	1.0053	1.0055	1.0057	1.0059	1.0053	1.0056	1.0058	1.0060
.20	1.0206	1.0217	1.0226	1.0235	1.0243	1.0217	1.0228	1.0238	1.0248
.30	1.0483	1.0510	1.0534	1.0557	1.0577	1.0511	1.0539	1.0566	1.0592
.40	1.0911	1.0969	1.1021	1.1069	1.1113	1.0972	1.1032	1.1093	1.1152
.45	1.1198	1.1280	1.1355	1.1423	1.1486	1.1285	1.1373	1.1460	1.1547
.50	1.1547	1.1664	1.1770	1.1867	1.1956	1.1672	1.1799	1.1926	1.2052
.55	1.1974	1.2140	1.2291	1.2429	1.2556	1.2153	1.2337	1.2524	1.2711
.60	1.2500	1.2739	1.2957	1.3155	1.3337	1.2760	1.3033	1.3312	1.3594
.65	1.3159	1.3510	1.3830	1.4121	1.4388	1.3546	1.3961	1.4392	1.4831
.70	1.4003	1.4534	1.5016	1.5456	1.5860	1.4600	1.5259	1.5957	1.6677
.75	1.5119	1.5955	1.6715	1.7409	1.8046	1.6094	1.7212	1.8427	1.9704
.80	1.6667	1.8099	1.9401	2.0589	2.1679	1.8407	2.0524	2.2901	2.5455
.85	1.8983	2.1732	2.4231	2.6513	2.8605	2.2614	2.7440	3.3121	3.9397
.90	2.2942	2.9548	3.5554	4.1042	4.6064	3.3327	4.9304	6.9353	9.2308

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TABLE II

RATIO OF MOMENTS FOR COMPRESSIBLE AND INCOMPRESSIBLE FLOWS

M_1	μ	M_c/M_i			
		$t' = 0.05$	$t' = 0.10$	$t' = 0.15$	$t' = 0.20$
0.10	1.0050	1.0051	1.0051	1.0050	1.0049
.20	1.0206	1.0207	1.0208	1.0206	1.0202
.30	1.0483	1.0486	1.0488	1.0478	1.0467
.40	1.0911	1.0920	1.0925	1.0914	1.0880
.45	1.1198	1.1212	1.1222	1.1208	1.1158
.50	1.1547	1.1570	1.1587	1.1570	1.1499
.55	1.1974	1.2012	1.2044	1.2023	1.1923
.60	1.2500	1.2564	1.2625	1.2605	1.2461
.65	1.3159	1.3274	1.3392	1.3385	1.3177
.70	1.4003	1.4222	1.4469	1.4511	1.4211
.75	1.5119	1.5580	1.6147	1.6358	1.5946
.80	1.6667	1.7789	1.9294	2.0135	1.9707
.85	1.8983	2.2438	2.7499	3.1211	3.1895
.90	2.2942	3.9315	6.5741	8.9709	10.4271

TABLE III

MOVEMENT OF CENTER OF PRESSURE AS FUNCTION OF STREAM

MACH NUMBER AND THICKNESS COEFFICIENT

M_1	μ	$\frac{C_c - C_i}{2a}$			
		$t' = 0.05$	$t' = 0.10$	$t' = 0.15$	$t' = 0.20$
0.10	1.0050	-0.0001	-0.0001	-0.0002	-0.0002
.20	1.0206	-.0002	-.0004	-.0007	-.0009
.30	1.0483	-.0006	-.0011	-.0018	-.0024
.40	1.0911	-.0011	-.0022	-.0034	-.0049
.45	1.1198	-.0015	-.0030	-.0047	-.0067
.50	1.1547	-.0021	-.0040	-.0063	-.0092
.55	1.1974	-.0028	-.0054	-.0085	-.0124
.60	1.2500	-.0039	-.0071	-.0113	-.0167
.65	1.3159	-.0048	-.0092	-.0149	-.0223
.70	1.4003	-.0062	-.0117	-.0193	-.0296
.75	1.5119	-.0076	-.0139	-.0239	-.0382
.80	1.6667	-.0080	-.0135	-.0257	-.0452
.85	1.8913	-.0019	.0005	-.0123	-.0381
.90	2.2942	.0427	.0750	.0624	.0259

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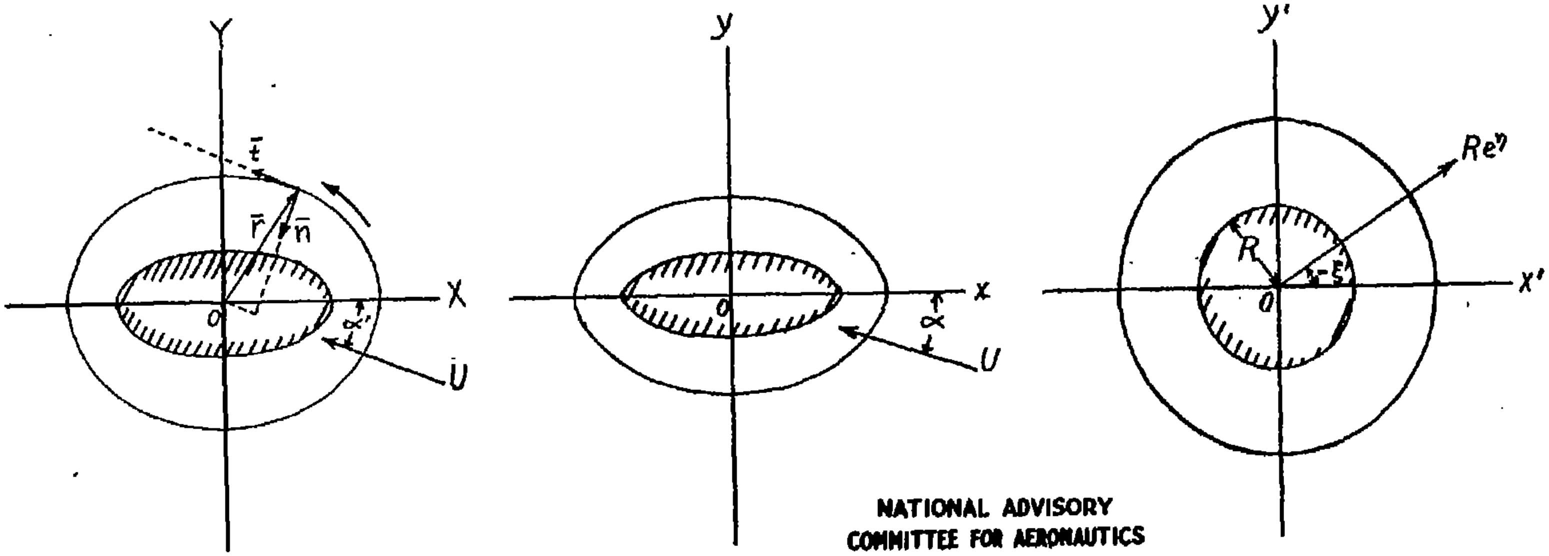
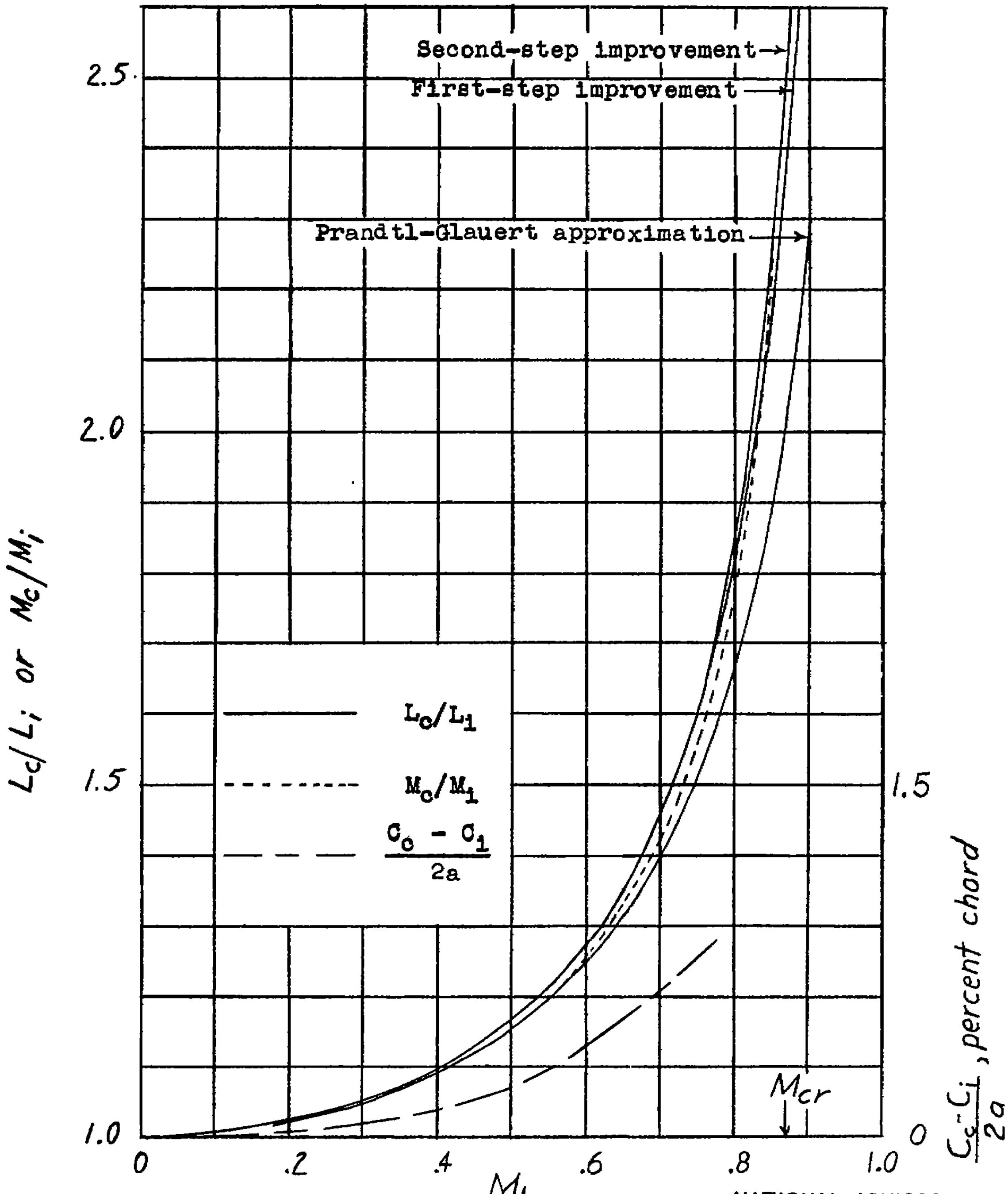


(a) Z-plane.

(b) z-plane.

(c) z'-plane.

Figure 1.- Flow plane Z with directions of tangent and normal on control contour; plane z of affinely distorted profile; plane z' of circle conformally related to profile in z'-plane.



(a) t' = 0.05.

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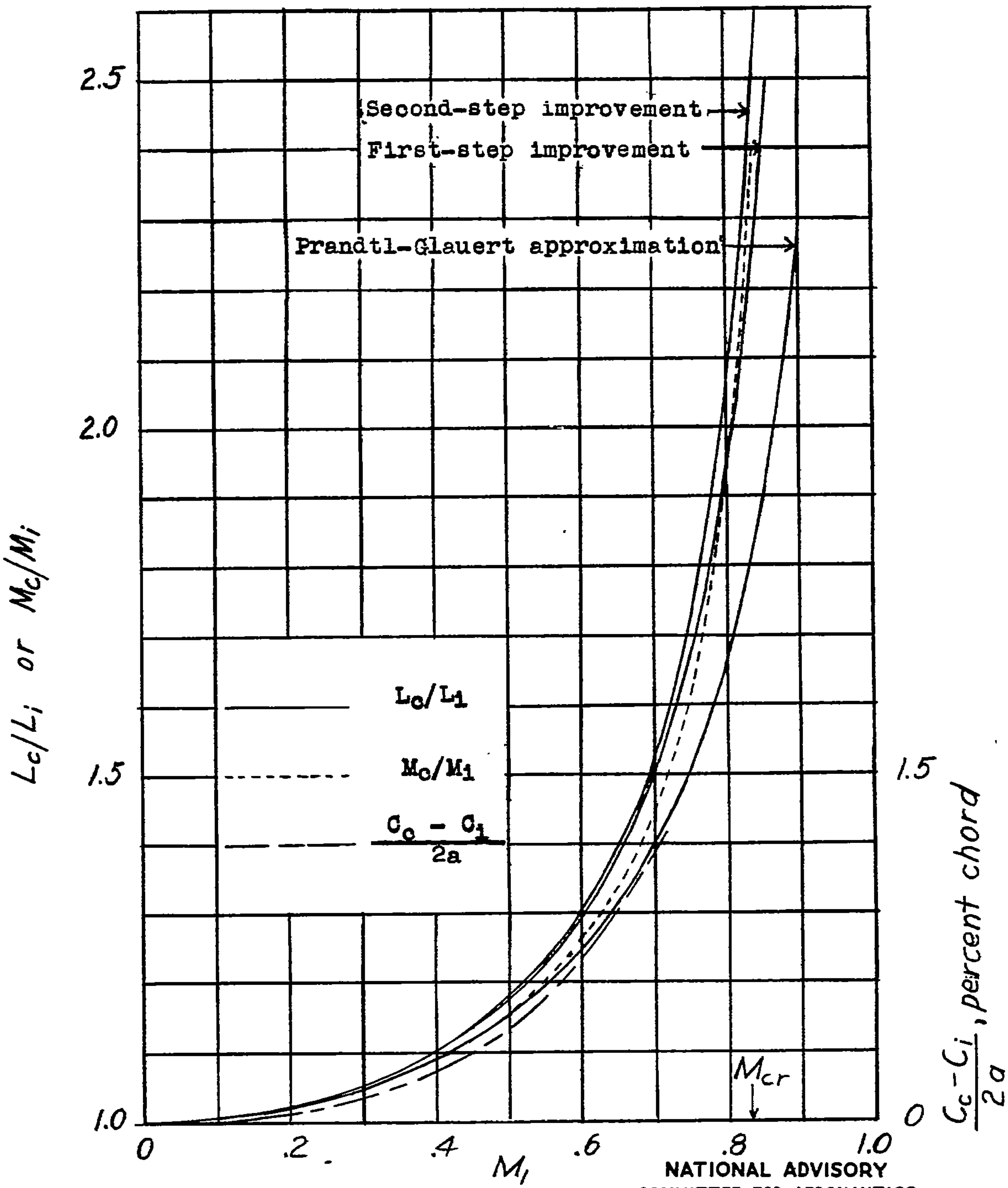
Figure 2.- Ratio of lifts and ratio of moments in compressible and incompressible flows and movement of center of pressure in percent chord as functions of stream Mach number. Center of pressure movement rearward with increasing stream Mach number.



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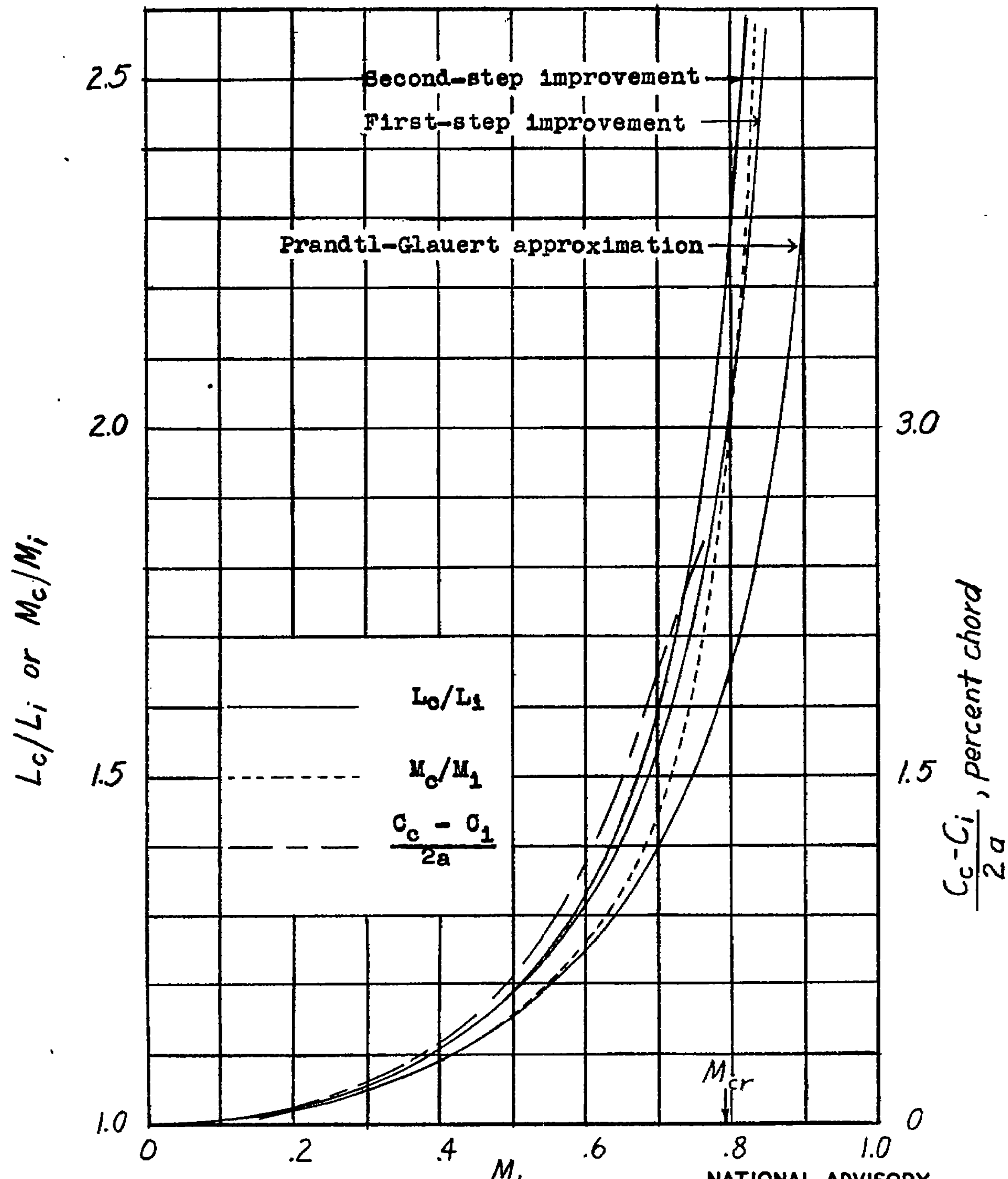
(b) t' = 0.10.

Figure 2.- Continued.



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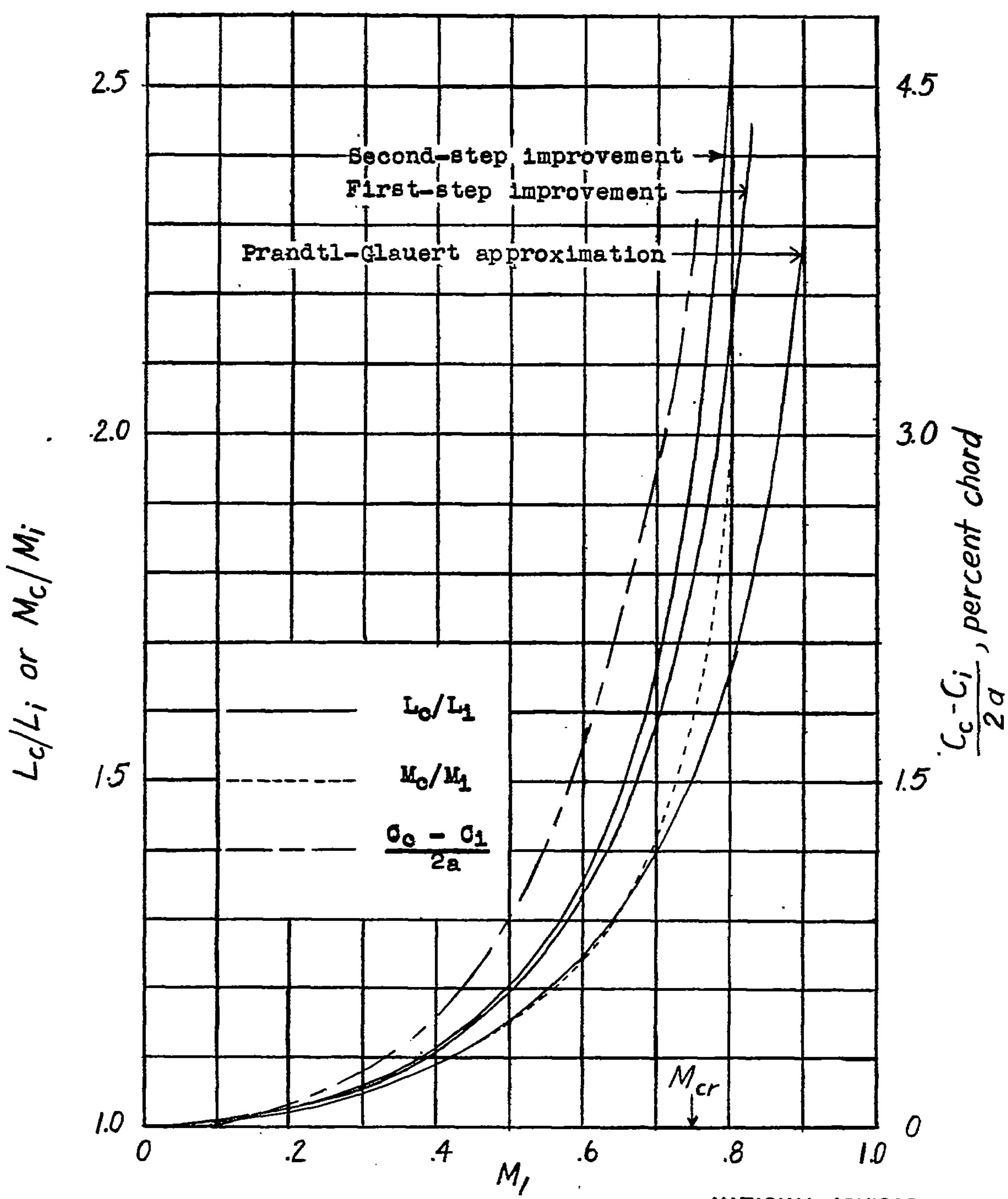
(c) t' = 0.15.

Figure 2.- Continued.



(d) $t' = 0.20$.

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Figure 2.- Concluded.